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GENERALIZED DYNAMIC PROBLEM OF THERMOELASTICITY FOR A HALF-SPACE HEATED BY LASER RADIATION*

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A generalized dynamic problem of thermoelasticity is solved for a half-space heated by laser radiation. Expressions for the displacements in the Rayleigh wave are obtained. The asymptotic form of the solution at a point at infinity is studied. It is shown that the magnitude of the displacements at the wave fronts depends essentially onthe value of the rate of propagation of heat.

1. Formulation of the problem. Let a beam of radiant energy fall, at the instant $\tau = 0$, on a circular region of a plane boundary of an elastic half-space. The position of every point of it is determined by the coordinates ρ, z, θ_1 of a cylindrical coordinate system. The radiation intensity volume density of the beam is

$$q_v(\rho, \tau) = q_1(\rho) H(\tau), \quad q_1(\rho) = \begin{cases} q_0, & 0 \leqslant \rho \leqslant R_0 \\ 0, & \rho > R_0 \end{cases} \tag{1.1}$$

($H(\tau)$ is Heaviside's function). We require to find the elastic stresses and displacements in the half-space when the radiant energy is absorbed. The variation in the temperature field caused by the deformation is ignorded.

The solution of this problem can be reduced to solving the following set of Eqs. /1/:

$$\left(\Delta - c_1^{-2} \frac{\partial^2}{\partial \tau^2}\right)\Phi = mt, \quad \left(\Delta - c_2^{-2} \frac{\partial^2}{\partial \tau^2}\right)\Psi = 0, \quad \left(\Delta - \frac{l}{a} \frac{\partial}{\partial \tau}\right)t = 0 \tag{1.2}$$

$$l = 1 - t_r \frac{\partial}{\partial \tau}, \quad m = \frac{3\lambda + 2\mu}{\lambda + 2\mu} \alpha_t$$

Here Φ, Ψ are the displacement potentials, t is temperature, c_1, c_2 are the velocities of the longitudinal and transverse wave, t_r is the thermal flux relaxation time, a is the thermal conductivity, λ, μ are the Lamè coefficients, α_t is the coefficient of thermal expansion, and Δ is the Laplace operator.

The solutions of the system must satisfy the following boundary and initial conditions:

$$\sigma_{zz} = \sigma_{\rho z} = 0, \quad -\lambda_q \frac{\partial t}{\partial z} = \eta l q_v \tag{1.3}$$

$$\Phi = \Psi = t = \frac{\partial t}{\partial \tau} = \frac{\partial \Phi}{\partial \tau} = \frac{\partial \Psi}{\partial \tau} = 0 \tag{1.4}$$

(σ_{ij} is the thermoelastic stress tensor, η is the absorption capacity and λ_q is the thermal conductivity.

2. Construction of the solution. We shall construct the solution of the problem using the contour-integral method /2/. Let us write the solution sought in the form of the Fourier-Bessel transform

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$$\Phi = \int_0^\infty \bar{A}_0(k, z)\, kJ_0(k\rho)\, dk, \quad \Psi = \int_0^\infty \bar{A}_1(k, z)\, kJ_1(k\rho)\, dk, \tag{2.1}$$

$$t = \int_0^\infty \bar{A}_2(k, z)\, kJ_0(k\rho)\, dk, \quad \bar{A}_j(k, z) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} A_j(p, k, z)\, e^{p\tau}\, dp,$$

$$j = 0, 1, 2$$

The unknown functions A_j are found by substituting relations (2.1) into (1.2)-(1.4) and solving the resulting ordinary differential equations in the same manner as in /2/. Let us write the final expressions for A_j

$$A_0 = T_1 e^{-\beta_1 z} + T_2 e^{-dz}, \quad A_1 = T_3 e^{-\beta_2 z}, \quad A_2 = \frac{B_1}{pkd} e^{-dz} \tag{2.2}$$

$$T_1 = \frac{T_2}{T_0}\left[(k^2 + \beta_2^2)^2 - 4k^2\beta_2 d\right], \quad T_2 = \frac{bc_1^2}{kp^2 d} \varphi_1(p)\, J_1(kR_0)$$

$$T_3 = \frac{2T_2}{T_0}(k^2 - \beta_2^2)(\beta_1 - d)\, k, \quad T_0 = 4k^2\beta_1\beta_2 - (k^2 - \beta_2^2)^2$$

$$a_0 = \frac{c_1^2}{a}, \quad a_1 = \frac{c_1^2}{c_q^2} - 1, \quad \beta_l^2 = k^2 - \frac{p^2}{c_l^2}, \quad l = 1, 2;$$

$$d^2 = k^2 - \frac{p}{a} + \frac{p^2}{c_q^2}$$

$$\varphi_1(p) = \frac{1}{a_0 + a_1 p}\left(1 - \frac{ap^2}{c_q^2}\right)$$

(c_q is the rate of heat propagation). The branches of the radicals in (2.2) are fixed by the condition that $\arg \beta_1 = \arg \beta_2 = \arg d = 0$ when $p > 0$.

We write the expressions for the components of elastic displacements in the form

$$u_\rho = \frac{1}{2\pi i} \int_0^\infty \left\{ \int_{\sigma - i\infty}^{\sigma + i\infty} \left[kT_1 e^{-\beta_1 z} - kT_2 e^{-dz} - \beta_2 T_3 e^{-\beta_2 z}\right] e^{p\tau}\, dp \right\} \times \tag{2.3}$$

$$kJ_1(kR_0)\, J_1(k\rho)\, dk$$

$$u_z = \frac{1}{2\pi i} \int_0^\infty \left\{ \int_{\sigma - i\infty}^{\sigma + i\infty} \left[kT_3 e^{-\beta_2 z} - T_1\beta_1 e^{-\beta_1 z} - dT_2 e^{-dz}\right] e^{p\tau}\, dp \right\} kJ_1(kR_0)\, J_0(k\rho)\, dk$$

Analysing expressions (2.2) we find that $A_j(p, k, z)$ are analytic functions of the complex variable p in the region (G: $\mathrm{Re}\, p > -a_0 a_1^{-1}$) when $c_q > c_1$, and in the region (G: $\mathrm{Re}\, p > 0$) when $c_q \leqslant c_1$. Analytic continuation $A_j(p, k, z)$ to the left half-plane is a multivalued function with branch points

$$p_{1,2} = \pm ikc_1, \quad p_{3,4} = \pm ikc_2, \quad p_{8,9} = -\frac{c_q^2}{2a} \pm \sqrt{\frac{c_q^4}{4a^2} - k^2 c_q^2}$$

and simple poles

$$p_{5,6} = \pm ikc_R, \; p_7 = -a_0 a_1, \; p_{10} = 0$$

We shall consider, on the upper sheet of the multisheeted Riemann surface the branch of the multivalued function $A_j(p, k, z)$ which represents the analytic continuation of this function, first defined in the region G. Every sheet of the Riemannian surface represents a plane p with cuts carried out as shown in Fig.1.

Following /3/, we shall represent the whole field of displacements in the form

$$U = U_0 + U_R + U_\lambda$$

where U_0 describes the static part of the problem and is determined by the contribution of the pole $p_{10} = 0$, U_R describes the Rayleigh wave and is determined by the contribution of the poles $p_{5,6} = \pm ikc_R$, representing the solution of the equation $T_0 = 0$, U_λ describes the volume waves and is obtained by integrating along the contour λ shown in Fig.1. We eliminate U_0 from further consideration, since we concern ourselves here only with the dynamic part of the problem.

3. Determination of the displacement field in the Rayleigh wave. If the deformation of the initial contour of integration into the contour λ is accompanied by intersection of the poles $P_{5,6}$, then the contribution of these poles determining the Rayleigh wave must be taken into account. Determining the residues at these poles we obtain

$$U_{\rho R} = \frac{bc_1^2 c_2^2}{c_R^3 c_0} \mathrm{Re}\left[\int_0^\infty \left\{\left(\frac{4\eta_1}{ix} - \frac{1 + \eta_1^2}{id_1}\right) e^{-\mu_1 x} + \eta_1(1 - \eta_1^2)\left(\frac{\eta_1}{ix} - \frac{\eta_2}{id_1}\right) e^{-\mu_2 x}\right\} \varphi_1(ix)\, J_1\left(\frac{xR_0}{c_R}\right) J_1\left(\frac{x\rho}{c_R}\right) dx\right] \tag{3.1}$$

$$U_{zR} = \frac{bc_2^2}{4c_R^3 c_0} \operatorname{Re}\left[\int_0^\infty \left\{\left(\frac{4\eta_2}{ix} - \frac{\eta_1^2+1}{id}\right) e^{-\mu_1 x}\eta_1 + \right.\right.$$

$$\left.\left.(1+\eta_1^2)\left(\frac{1}{ix} - \frac{\eta_1}{id}\right)e^{-\mu_2 x}\right\} \varphi_1(ix) J_1\left(\frac{xR_0}{c_R}\right) J_0\left(\frac{x\rho}{c_R}\right) dx\right]$$

$$d_1 = \sqrt{x(ia_3 + a_2 x)}, \quad a_2 = \frac{c_R^2}{c_q^2} - 1, \quad a_3 = \frac{c_R^2}{u}$$

$$\eta_k^2 = \frac{c_k^2 - c_R^2}{c_k^2}, \quad \mu_k = \frac{z\eta_k}{c_R} + i\tau, \quad k = 1, 2$$

$$c_0 = \frac{\eta_2^2 - \gamma^2\eta_1^2}{\eta_1\eta_2} - \eta_1 - 1, \quad \gamma = \frac{c_2}{c_1}$$

Let us consider a special case of a point source obtained from (3.1) by the following passage to the limit:

$$\lim_{R_0 \to 0} U_{\rho R} = U_{\rho R}^c, \quad \lim_{R_0 \to 0} U_{zR} = U_{zR}^c$$

where we have (W is the power of the optical radiation source)

$$\lim_{R_0 \to 0} bR_0 = b_1 = \frac{\eta W m}{\lambda_q}$$

Since it is not possible to derive an expression for the displacement components in closed form, we shall attempt to obtain the approximate expressions for $U_{\rho R}^c$ and U_{zR}^c. Let us consider the integral

$$\Phi(\lambda_2) = \int_0^\infty t^\nu e^{-\lambda_2 t} J_1(\lambda_1 t) f(t)\, dt \tag{3.2}$$

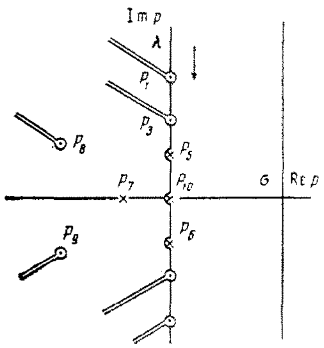

Fig 1.

Lemma. Let

$$f(t) \in c^\infty([0;\infty]); \ \operatorname{Re}\lambda_2 > 0, \ \nu > -2$$

Then the following asymptotic expansion, as $|\lambda_2| \to \infty$, holds:

$$\Phi(\lambda_2) = \Sigma \frac{f^n(0)}{n!}\frac{\lambda_1}{2}\Gamma\left(\frac{\nu_1}{2}\right)\frac{1}{(\lambda_1^2+\lambda_2^2)^{\nu_1}} \times F\left(\nu_1, \frac{2-n-\nu}{2}, 2, \frac{\lambda_1^2}{\lambda_1^2+\lambda_2^2}\right), \quad \nu_1 = \frac{2+n-\nu}{2} \tag{3.3}$$

($F(a, b, c, z)$ is the hypergeometric function and $\Gamma(z)$ is the gamma function).

Proof. We take $t = t_0$ such that $t_0 < 1$. Then

$$\left|\int_{t_0}^\infty t^{\nu-n} e^{-\lambda_2 t} J_1(\lambda_1 t)\, dt\right| < \int_{t_0}^\infty |t^{\nu-n} e^{-\lambda_2 t}|\, dt < \int_{t_0}^\infty t^{n-\nu} e^{-\operatorname{Re}(\lambda_2 t)}\, dt = F_1(\lambda_2)$$

When $\operatorname{Re}\lambda_2 > 0$, the last integral has the following estimate by virtue of Lemma 1.1 of /4/:

$$F_1(\lambda_2) < c \exp[\operatorname{Re}(\lambda_2 t_0)]$$

Let us consider the integral

$$\Phi_n(\lambda_2) = \int_0^{t_0} t^{n-\nu} e^{-\lambda_2 t} J_1(\lambda_1 t)\, dt$$

Let us write $\Phi_n(\lambda_2)$ in the form of a difference of integrals along the semiaxes $[0, \infty)$ and $[t_0, \infty)$. Then the first integral can be found from the tables of integration /5/, and for the second integral the estimate obtained above holds. Expanding the function $f(t)$ on the segment $[0, t_0]$ in a uniformly converging Maclaurin's series and integrating the resulting expression term by term, we arrive at formula (3.3).

Using the substitution $t = a_0^{-1}x$, we reduce the integrals appearing in (3.1) to the form (3.2), and $\lambda_2 = a_0\mu_k$, $k = 1, 2$. It should be noted that for most materials the quantity $a_0 = c_1^2 a_0^{-1}$ is of the order of 10^{12}, and $\operatorname{Re}\lambda_2 \sim 10^6 z$, and this enables us to use the asymptotic expansion (3.3) with an accuracy, sufficient in practice for any, even very small values of z. Restricting ourselves to the principal term of the expansion, we shall write the following expressions for the displacements:

$$U_{\rho R}^c = b_2 \operatorname{Re}[Q_1 - P_1 - \eta_1(\eta_1^2 - 1)Q_2 - \eta_1\eta_2 P_2]$$

$$U_{zR}^c = b_2 \operatorname{Re}\left[\eta_2 Q_1 - \eta_1 P_1 - \frac{\eta_1^2+1}{\eta_1} Q_2 - \eta_1 P_2\right]$$

$$Q_k = 4c_R\eta_1 \frac{\sqrt{(\mu_k c_R)^2 + \rho^2} - \mu_k c_R}{\rho\sqrt{(\mu_k c_R)^2 + \rho^2}}$$

$$P_k = \frac{3}{16\rho}\sqrt{\frac{\pi}{a_3}} c_R^3 (\eta_1^2 + 1) \quad [(\mu_k c_R)^2 + \rho^2]^{-5/4}$$

$$F\left(\frac{5}{4}, \frac{3}{4}, 2, \frac{\rho^2}{(\mu_k c_R)^2 + \rho^2}\right), \quad k = 1, 2, \quad b_2 = \frac{b_1}{4a_0(\eta_1^2 + 1)c_0}$$

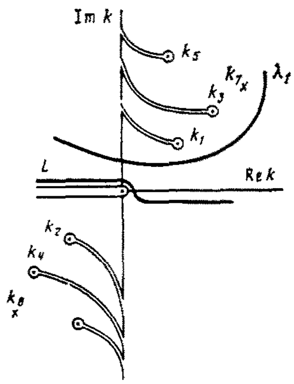

Fig.2

4. Asymptotic form of U_λ as $R \to \infty$ $R = \sqrt{\rho^2 + z^2}$.

We fix the branches of the radicals in (2.2) by means of the condition $\operatorname{Re}(\beta_1, \beta_2, d) > 0$. This enables us to change the order of integration in (2.3), provided that the path of integration in the complex variable plane p coincides with the path λ (Fig.1), passing along the imaginary axis. Considering the integrals in question in the plane of the complex variable k, we shall write the solution in another form which will take, e.g. for the first term of (2.1), the form

$$\Phi = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\left[\int_L A_0(p, k, z)\, k H_0(k\rho)\, dk\right] e^{p\tau}\, dp$$

The path of integration L in the plane of the complex variable k is shown in Fig.2. The singularities indicated in Fig.2 are

$$k_{1,2} = \pm i\frac{p}{c_1}, \quad k_{3,4} = \pm i\frac{p}{c_2}$$

$$k_{5,6} = \pm\sqrt{\frac{p}{a} + \frac{p^2}{c_q^2}}, \quad k_{7,8} = \pm i\frac{p}{c_R}$$

To illustrate the geometrical constructions, all singularities of the integrands are removed from the real axis.

Deforming the initial path L into the path λ_1, coinciding with the path of steepest descent, we obtain the asymptotic form of the solution using the method of steepest descent

$$U_l = \frac{bc_1^2}{2\pi R}\int_{-i\infty}^{i\infty}\left[4p^{-1}\gamma^2\sin^2\theta\sqrt{1 - \gamma^2\sin^2\theta} + \frac{(1 - 2\gamma^2\sin^2\theta)^2}{\sqrt{p\,[a_0 + (a_1 + \cos^2\theta)\,p]}}\right]\frac{\operatorname{tg}\theta\,\varphi_1(p)}{T_l}\cdot J_1(i\gamma_1 p)\, e^{p\tau_1}\, dp \tag{4.1}$$

$$U_t = \frac{bc_2^2}{2\pi R}\int_{-i\infty}^{i\infty}\frac{\cos 2\theta}{p} - \frac{\sqrt{\gamma^2 - \sin^2\theta}}{\sqrt{p\,[a_0\gamma^2 + (a_4 + \cos^2\theta)\,p]}}\cos^2\theta\; \varphi_1(p)\, J_1(\gamma_2 p)\, e^{p\tau_2} dp \tag{4.2}$$

where

$$U_l = \frac{U_{\rho l}}{\sin\theta} = \frac{U_{zl}}{\cos\theta}, \quad U_t = \frac{U_{\rho t}}{\cos\theta} = \frac{U_{zt}}{\sin\theta}, \quad \theta = \operatorname{arctg}\frac{\rho}{z}$$

$$T_l = (1 - 2\gamma^2\sin^2\theta)^2 + 4\gamma^3\sin^2\theta\cos\theta\sqrt{1 - \gamma^2\sin^2\theta}$$

$$T_t = \cos^2 2\theta + 4\sin^2\theta\cos\theta\sqrt{\gamma^2 - \sin^2\theta}$$

$$a_4 = \frac{C_2^2}{c_q^2}, \quad \gamma_j = \frac{R_0\sin\theta}{c_j}, \quad \tau_j = \tau - \frac{R}{c_j}, \quad j = 1, 2$$

and $U_{\rho l}, U_{zl}, U_{\rho t}, U_{zt}$ are the corresponding terms of the displacement field in (2.3), describing the fields of the longitudinal and transverse waves.

Let us now denote by U_q the part of the displacement field in (2.3) which describes the elastic wave propagating with a velocity equal to the velocity of heat propagation c_q. The asymptotic expression for U_q as $R \to \infty$ will be

$$U_q = b_2 c_1^2\int_{-i\infty}^{i\infty}\frac{J_1(id_0 R_0\sin\theta)}{p^2}e^{-d_0 R + p\tau}\varphi_1(p)\, dp, \quad d_0^2 = \frac{p}{a} + \frac{p^2}{c_q^2}$$

In the case of a point source we have

$$U_l = \frac{b_1 c_1}{RT_l a_0}\left\{\left[2a_1^{-1}\gamma^2\sin\theta\sin 2\theta\sqrt{1 - \gamma^2\sin^2\theta}\times\left(H(\tau_1) - \exp\left(-\frac{a_0}{a_1}\tau_1\right)\right) + (1 - 2\gamma^2\sin^2\theta)^2\times\frac{\cos\theta}{a_0\sqrt{a_1 - \cos^2\theta}}\Phi_1(\tau_1) - \frac{1}{a_1}\exp\left(-\frac{a_0}{a_1}\tau_1\right)\right)\right\}$$

$$U_t = \frac{b_1 c_2}{T_t a_0 R}\left[\frac{\sin 4\theta}{4}H(\tau_2) - \frac{\sqrt{\gamma^2 - \sin^2\theta}}{2a_0\sqrt{a_1 + \cos^2\theta}}\sin 2\theta\,\Phi_2(\tau_2)\right]$$

$$U_q^1 = \frac{b_1 c_1}{R a_0}\left[\exp\left(-\frac{a_0}{a_1}\tau_3\right) + \Phi_3\left(\tau, \frac{R}{c_q}\right)\right], \quad c_q \leqslant c_1$$
$$U_q^1 = \frac{b_1 c_1}{R a_0}\left[\exp\left(-\frac{a_0}{a_1}\tau_4\right) + \Phi_3\left(\tau, \frac{R}{c_q}\right), \quad c_q > c_1\right.$$

where

$$\Phi_{1,2} = \int_0^{a_{5,6}} \frac{e^{-x\tau_{1,2}}}{\sqrt{x(a_{5,6}-x)}}\varphi_1(-x)\,dx$$

$$\Phi_2\left(\tau, \frac{R}{c_q}\right) = \int_0^{a_7} \frac{\exp(-d_0 R - x\tau)}{\sqrt{x(a_7-x)}}\varphi_1(-x)\,dx$$

$$a_5 = \frac{a_0}{a_1 - \cos^2\theta}, \quad a_6 = \frac{a_0\gamma^2}{a_4 - \cos^2\theta}$$

$$a_7 = \frac{c_q^2}{a}, \quad \tau_3 = \tau - \frac{R}{c_q}, \quad \tau_4 = \tau - \frac{R}{c_q}$$

A study of the propagation of the discontinuities in the elastic displacement field is of interest. The discontinuities in the displacement field are caused by unequal convergence of the integrals (4.1)–(4.2) at the limit at infinity. Let us denote by W_l and W_t the magnitudes of the jumps at the longitudinal and trasnverse wave fronts corresponding to the discontinuities in question. From (4.1), (4.2) we obtain

$$W_l = \frac{b_1(a_1-1)c_1}{2T_l a_1 a_0 R}\left[4\gamma^3\sin^2\theta\sqrt{1-\gamma^2\sin^2\theta} - \frac{(1-2\gamma^2\sin^2\theta)^2}{\sqrt{a_1-\cos^2\theta}}\right]\cos\theta$$
$$W_t = \frac{b_1(a_1-1)c_2}{2T_t a_1 a_0 R}\left[\cos\theta - \sqrt{\frac{\gamma^2-\sin^2\theta}{a_1-\cos^2\theta}}\right]\sin 2\theta$$

Similarly, the discontinuities in the displacements field U_q^1 are described by the expression

$$U_q^1 = \frac{b_1(a_1-1)c_1}{2a_1 a_0 R}\exp\left(-\frac{c_q^2}{a}R\right)$$

Figure 3 shows the results of numerical computation of the normalized direction functions

$$F_1(\theta) = W_l/l\max$$

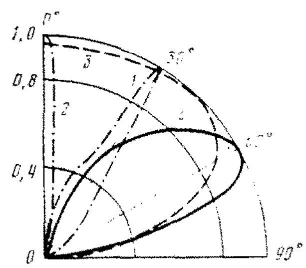

Fig.3

for various values of the quantity $\gamma_3 = \sqrt{a_1 - 1}$. The curves 1–4 correspond to the values $\gamma_3 = 0.5;\ 0,01;\ 5;\ 10$.

Thus the direction functions $F_1(\theta)$ depends substantially on the numerical value of the rate of heat propagation c_q. The problem concerning the quantity c_q can be solved after determining the direction function experimentally.

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